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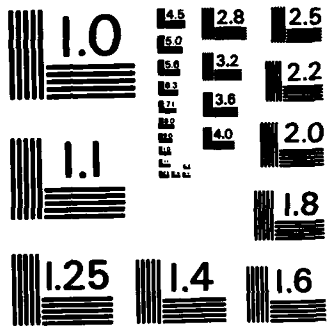

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ABSTRACT

A characterization of independence via uncorrelatedness is shown to hold for a family exhibiting positive dependence, wider than the one containing associated random variables. This generalizes a result of Newman and Wright (1981 Ann. Prob.) for associated random variables. In fact the weakening of the positive dependence condition naturally leads to a more direct and simple proof of the characterization. It also yields the same characterization for a family possessing an analogous negative dependence condition.

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Introduction. Among various notions of positive dependence, that of *association*, introduced by Esary, Proschan and Walkup (1967), (written as EWP henceforth), has proved to be quite useful. To define this concept, let $\underset{\sim}{X} = (X_1,\ldots,X_k)$ be a vector of real random variables. The vector $\underset{\sim}{X}$ (equivalently, components X_i) is (are) said to be associated if for every pair of functions f,g defined on $R^n \to R$, both nondecreasing (nonincreasing)

(1) $\qquad \operatorname{cov}[f(\underset{\sim}{X}),g(\underset{\sim}{X})] \geq 0,$

or equivalently, if one is nondecreasing and the other nonincreasing, then

(1A) $\qquad \operatorname{cov}[f(\underset{\sim}{X}),g(\underset{\sim}{X})] \leq 0.$

Here a function is said to be nondecreasing (nonincreasing) if it is so in each argument separately. In EWP it was shown that in particular, if $X_1,\ldots,X_k$ are independent random variables then they are associated. Recently, Newman and Wright (1981) showed that if $\underset{\sim}{X}$ is associated and $\operatorname{cov}(X_i,X_j) = 0$, for every pair (i,j) with $i \neq j$, then the X_i are independent. Their approach was designed for obtaining certain bounds for the difference between the characteristic function of $\underset{\sim}{X}$ and the product of characteristic functions of the X_i. Hence, as a proof of the above result, their approach is certainly not elementary. Newman and Wright (1981) also mention a proof of Wells (1977) which seems to be even more complicated.

The purpose of this note is two-fold:

a) To widen the applicability of the characterization. Since the condition of association is rather strong, it is desirable to look for a weaker condition of positive dependence for which uncorrelatedness would imply mutual independence. In fact we present an argument which shows that the dependence condition considered here is minimal in some sense. Further it provides a natural analog for negative dependence for which the same characterization holds.

b) To obtain a direct and elementary proof of the above characterization.

Regarding the negative dependence, a concept of <u>negative association</u> is studied in [2]. The random vector $\underset{\sim}{X}$ is said to be negatively associated if the reverse inequality holds in (1) (or (1A)), where now f, g are defined on <u>disjoint subsets</u> of $X_1,\ldots,X_k$. Our results will imply that negatively associated uncorrelated random variables are mutually independent.

<u>The Result</u>: Let $\underset{\sim}{X} = (X_1,\ldots,X_k)$ be a vector of k real random variables.

<u>Definition</u> 1. The distribution of $\underset{\sim}{X}$ (or $\underset{\sim}{X}$ itself) is said to be <u>strongly positively orthant dependent</u> (SPOD) if for an arbitrary subset A of the index set $\{1,2,\ldots,k\}$ and a vector of constants $\underset{\sim}{c} = (c_1,\ldots,c_k)$, the following conditions hold.

(2) $$P[\underset{\sim}{X} \geq \underset{\sim}{c}] \geq P[X_i \geq c_i,\ i \in A]P[X_j \geq c_j,\ j \in \bar{A}].$$

(3) $$P[\underset{\sim}{X} \leq \underset{\sim}{c}] \geq P[X_i \leq c_i,\ i \in A]P[X_j \leq c_j,\ j \in \bar{A}].$$

and

(4) $$P[X_i \geq c_i,\ i \in A;\ X_j \leq c_j,\ j \in \bar{A}]$$

$$\leq P[X_i \geq c_i,\ i \in A]P[X_j \leq c_j,\ j \in \bar{A}],$$

where $\bar{A}$ is the complement of A.

Remark 1. The dependence expressed by SPOD is stronger than positive upper orthant dependence (PUOD) which requires

$$(5) \qquad P[\underset{\sim}{X} \geq \underset{\sim}{c}] \geq \prod_{i=1}^{k} P[X_i \geq c_i],$$

and positive lower orthant dependence (PLOD) requiring

$$(6) \qquad P[\underset{\sim}{X} \leq \underset{\sim}{c}] \geq \prod_{i=1}^{k} P[X_i \leq c_i].$$

If (5) and (6) are both satisfied, the dependence is labeled as POD. For the bivariate case however, all five conditions (2) - (6) are equivalent and the dependence is called positive quadrant dependence (PQD), which was studied in detail by Lehmann (1966). On the other hand, if $\underset{\sim}{X}$ is associated then it is SPOD. This is easily seen by choosing f and g as products of the indicators of appropriate sets and applying either (1) or (1A). It is well known that even in the bivariate case, the association condition is strictly stronger than PQD.

Theorem: If $\underset{\sim}{X}$ is SPOD with uncorrelated components then the X_i are mutually independent.

Proof: Let $Y_i = I[X_i \geq c_i]$, $i = 1,\ldots,k$; where I is the indicator function. Since $\underset{\sim}{c}$ is arbitrary, it suffices to show that the Y_i are independent.

Now the Y_i are binary random variables and inherit SPOD from $\underset{\sim}{X}$. Further, SPOD of $\underset{\sim}{X}$ implies that every pair X_i, X_j is PQD. Lehmann (1966) showed that PQD together with uncorrelatedness implies independence of X_i, X_j and hence the Y_i are pairwise independent.

To motivate our proof we will establish the result for $k = 3$. The same technique together with induction yields the general result.

Let $p_i = P[Y_i = 1]$ and $P[1,1,0]$ be the probability of $Y_1 = 1$, $Y_2 = 1$, $Y_3 = 0$ and so on.

From (4) and pairwise independence, it follows that

$$(7) \qquad P[1,0,1] \leq p_1(1 - p_2)p_3.$$

In general, a similar inequality holds whenever a triplet contains both 0 and 1. For example,

$$(8) \qquad P[0,0,1] \leq (1 - p_1)(1 - p_2)p_3.$$

However, these have to be <u>equalities</u>, because if not, they would imply (by adding)

$$(9) \qquad P[Y_2 = 0,\ Y_3 = 1] < (1 - p_2)p_3,$$

violating the pairwise independence.

The only terms with possible reverse inequalities (apply (2) and (3)) are

$$(10) \qquad P[1,1,1] \geq p_1p_2p_3$$

and

$$(11) \qquad P[0,0,0] \geq (1 - p_1)(1 - p_2)(1 - p_3).$$

But again these have to be equalities since the sum of the right sides of other terms is 1 and cannot be exceeded by the sum of the left sides.

For the induction step, one may assume that every subset of cardinal-

random variables which are mutually independent. This will lead to inequalities similar to (7) or (8) for every k-tuple having both a 0 and a 1. The rest of the argument is similar. //

Let $\underset{\sim}{X}$ be called strongly negatively orthant dependent (SNOD) if the inequalities separating the left and right sides in (2), (3) and (4) are reversed. Note that the negative association defined in the introduction implies SNOD.

Corollary: $\underset{\sim}{X}$ is SNOD with uncorrelated components implies the X_i are mutually independent.

Remark 2. For any notion of positive dependence which transmits those conditions to the indicators Y_i defined above, the characterization of independence will have to hold for these binary variables. If the inequalities such as (9) or (10) do not go in the same direction one could assign probability mass such that all others are equalities while the mutual independence fails because of those terms. In this sense, the inequalities defining the positive (negative) dependence seem to be necessary.

Finally, consider the classical Bernstein example where a tetrahedron has 3 sides with 3 distinct colors and the fourth has stripes of all three. If X_i denotes the indicator of the presence of the ith color at the bottom of the tetrahedron (after a toss) then it is well known that the X_i's are pairwise independent but not mutually independent. It is interesting to note that the X_i's are (strictly) PUOD as well as NLOD. This illustrates that weak positive and negative dependence may hold at the same time, and in spite of pairwise independence, the mutual independence might fail.

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